18.152 PROBLEM SET 2

due February 26th 9:30 am

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *Let $\Omega \subset \mathbb{R}^n$ be a convex open set with smooth boundary $\partial\Omega$, and let $w : \overline{\Omega} \to \mathbb{R}$ be a smooth function satisfying $\Delta w = 0$. Suppose that a smooth solution $u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ to the heat equation satisfies $u(\vec{x},t) = w(\vec{x}) = g(\vec{x})$ on $\partial\Omega \times [0,T]$ and $u(\vec{x},0) = g(\vec{x})$ for a smooth function g. Establish upper bounds for $|u|$ and $\|\nabla u\|$ in terms of g, w, and their derivatives.*

Problem 2. *Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary $\partial\Omega$. Suppose that there are two smooth solutions u and v to the heat equation defined over $\overline{\Omega} \times [0,T]$ such that $u(\vec{x},T) = v(\vec{x},T)$ holds in $\overline{\Omega}$ and $u_\nu(\vec{x},t) = v_\nu(\vec{x},t)$ holds on $\partial\Omega \times [0,T]$, where ν denote the outward pointing unit normal vector of $\partial\Omega$. Show that $u(\vec{x},t) = v(\vec{x},t)$ holds on $\overline{\Omega} \times [0,T]$.*

Problem 3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with smooth boundary $\partial\Omega$ and the unit volume $|\overline{\Omega}| = \int_{\overline{\Omega}} d\vec{x} = 1$. Suppose that there is a smooth solution $u : \overline{\Omega} \times [0,1)$ to the semilinear heat equation*

$$u_t = \Delta u + u^2.$$

Moreover, $\int_\Omega u(\vec{x},0)d\vec{x} = 1$ and $u_\nu(\vec{x},t) = 0$ holds on $\partial\Omega \times [0,1)$, where ν is the outward pointing unit normal vector to $\partial\Omega$. Show

$$\int_{\Omega} u(\vec{x},t)d\vec{x} \geq \frac{1}{1-t}.$$

Problem 4. *Suppose that a positive smooth function $u : \mathbb{R}^n \times (-\infty, T] \to \mathbb{R}$ is a solution to the heat equation. Moreover, for each $i \in \{1, \cdots, n\}$ we have $u(\vec{x},t) = u(\vec{x} + \vec{e}_i, t)$ where $\vec{e}_1 = (1,0,\cdots,0)$, $\vec{e}_2 = (0,1,0,\cdots,0)$, $\cdots$, $\vec{e}_n = (0,\cdots,0,1)$. Show that $u_t \geq 0$.*